**RSTDPT (Constant)**

There are a few nontrivial time-dependent examples that can be done exactly. One such is the delta function potential. We already solved the time-independent case. Now let’s do the time-dependent one. Hold on to your butts.

**Example**

Let’s specialize to a delta function potential.



We’d like to see how much progress we can make calculating its time-dependence. We’ll start with the GF, of course.



Turns out it most useful to consider G in terms of T. So we wrote that:



and T itself is given by:



We’d like to solve for T and then for G. The best way to solve for T is via a recursion relation (trying to sum infinite series isn’t typically the way we want to proceed). Well, from that infinite series, we see that we can write,



And now let’s put this in a basis. We’ll dot both sides with <k| and |k´>, and insert resolutions of identity as needed,



Now the nice thing about the δ potential is that



So we have:



Pondering this equation a bit, we might observe that on the right hand side, the k1 index of Tk1k´ is integrated over, meaning the result of the integral will only depend on the k´ index. Thus considering Tkk´(ω) is equal to this thing, that means that Tkk´(ω) can at most depend on its k´ index alone. Given this, we can eliminate the first index of T from its argument, and write,



which makes it now trivial to solve for T,



But we’ll now observe that the RHS doesn’t even depend on k´. So really, T doesn’t have any index dependence at all – that’s what’s nice about the delta function potential. So we have, changing variable of integration from k1 -> k:



Let’s now do this integral,



If ω > 0, we can factor the denominator, and close the contour, up, say, to get:



And if ω < 0, then we do similarly, and close the contour up, say,



If we agree that √(-|ω|) = i√ω, then we can combine these two results to simply say,



Yay! We’ve calculated the T-matrix exactly. With this, we can get the GF,



So we have:



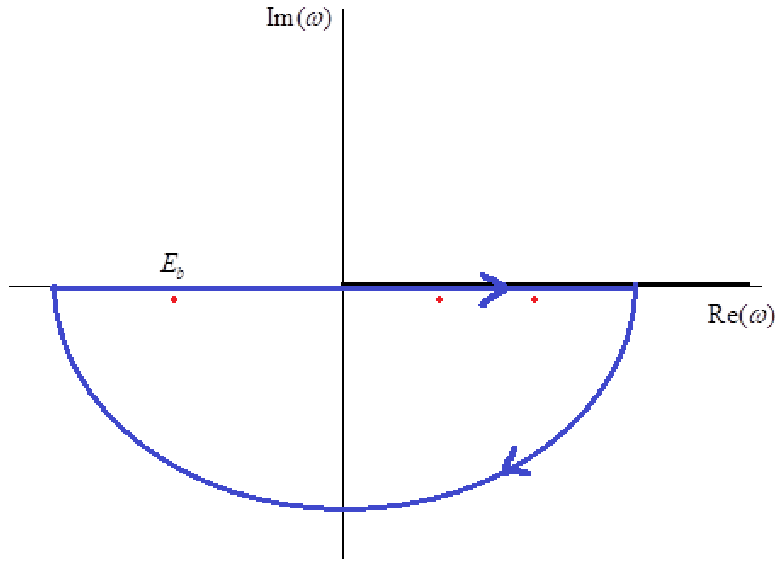
Let’s note that as argued, Gkk´(ω) has a branch cut along the Re(ω) > 0 axis, thanks to the presence of the √ω in there. And also, it has a pole at the bound state energy. Recall from the time-independent file that we found for V0 < 0, there was a single bound state with energy εb = -(1/2)mV02. This should show up as a pole, i.e., singularity of Gkk´(ω). And it does.



(if plug this back into the top line, can see this solution only works if V0 < 0) And we’ll also observe Gkk´ has poles at ω = εk, the free particle scattering states spectrum. Let’s see if we can get Gkk´(t). So we’ll need the inverse Fourier transform of G,



Again, we’ll have to resort to Complex Analysis/Residue tricks to do the integrals. So we will use a semicircular contour, closed down. And we’ll run clockwise around the contour, with a semicircular bump around ω = εb too. Actually, I’m goint to shift the pole to ω = εb – iη, because I think all poles need to be below Re(ω) to preserve fact that G(t) is proportional to θ(t). Hmm…. The branch cut is in dark black, and the two red dots right below it are the poles at ω = εk – iη, and εk´ - iη. Looks like this:



So, doing out integral,



And proceeding, and using the formula that for a first order pole, we have:



So,



So I guess we have:



Therefore we have Ukk´(t), and Skk´(t),



what is limit as k´ 🡪 k? In the long time limit, this was supposed to go to a delta function. So let’s see. Supposing k´ ≠ k, then,



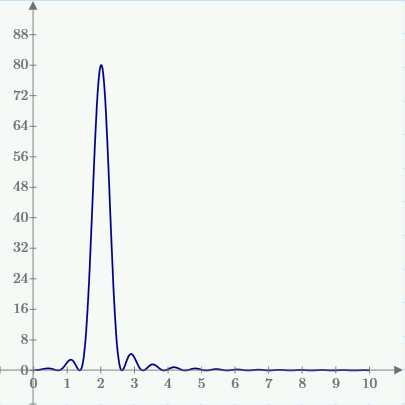
and,



This matches the general form expected from PT. What happens in the large t limit? Well, let’s do the modulus,



I plotted the modulus |Skk´(t)|2 as a function of k´ below, for t = 10, k = 2, m = 1, ℏ = 1, V0 = -1, and it looks pretty delta-functiony, indicating that the probability of transition from k´ to k in the long time limit is nearly zero unless the energies εk, εk´ are the same. That’s encouraging.



And now to get the transition rate, we’d differentiate:



Using limt🡪∞ sin(xt)/πx = δ(x), this comes to, neglecting the second term as being much smaller in this limit,



as we found it should be, in a previous file. So we’ve in principle solved this problem. For instance, supposing we have a particle in an initial state ψ0(x), and we want to know its time-development there after, we’d just do:



which we can write as:



We will recognize Uxx´(t) as the ‘propagator’ when we get to the path-integral formulation of dynamics.

**Simplifying the Propagator**

Okay we’ll let’s try to simplify Uxx´(t) for our problem,



We’ll split this up into three parts. The first guy is the ‘free’ particle contribution,



And next, the middle parts,



We’ll note the second term is the same as the first term, just with x and x´ switched (just making change of variables k -> k´, k´ -> k, and k -> -k, k´ -> -k´). We’ll also parenthetically note that we can see Uxx´(3)(t) is a function of |x|, and |x´|, as making transformation x -> -x, and x´ -> -x´ leaves integral invariant (just make change of variables k -> -k, and k´ -> -k´ to see this). Well, let’s just focus on the first term, which we’ll call Uxx´(3a).



where we recall we argued each piece was a function of |x| and |x´| separately. Now before we continue, let’s combine the pieces. Recall,



I don’t entirely understand this next step. So √(2mεk) = √(k2) is to be interpreted not as |k|, but rather as an analytic function, k. This kind of makes sense, as if we wrote in terms of modulus and phase, we’d have: √(|k|2e2iφ) = |k|eiφ = k; I’m just not *entirely* convinced we should do this. Nonetheless, if we don’t do this, then we don’t get the right answer (or any answer – as I can’t figure out the resulting integral), so ….



In the following steps, we’ll employ Feynman’s trick of differentiating and integrating w/r to a parameter to simplify the integrand,



This result is undefined until we have specified a lower bound for the integral. I think we can use μ = -∞. This should work, as if |x| + |x´| were to go to -∞ (pretending for moment that it ‘could’ be negative), then our dμ integral would go to zero, which would make our Uxx´(3) term go to zero likewise. And our Uxx(3)­ guy *should* go to zero when this happens as can see from



as the large oscillations brought about by the eik(|x|+|x´|) term would wash out the integral. So let’s write,



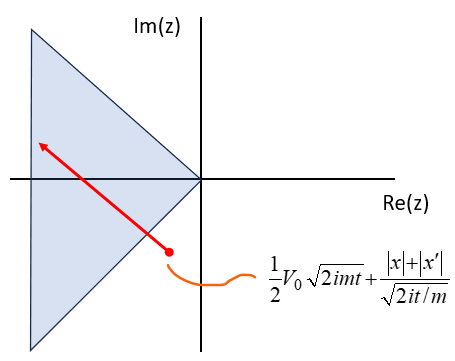
Now we switch order of integration,



Now we can do the ∫dμ integral. It is an error function, or can be related to one.



The integral contour would be something like this, remembering V0 < 0:



(the blue triangle is the region of space in the LHS within which the integral would converge) So we have:



The last part comes from the bound state our δ function potential has,



Might observe this is exactly the time-development of a delta function bound state, as can verify by referencing the bound state eigenfunction we calculated in the time-independent file. So altogether, it appears we have:



Got a problem though. Say V0 🡪 -∞, then using the asympotic expansion of the erfc, erfc(z) ~ exp(-z2)/z√π, we have:



Let’s compare to what Uxx´ ought to be for particle confined to half-line (which is what we’d have if the potential at the origin were indeed infinite):



Our δ result should reduce to this when V0 -> ∞. So looks like I have missed a factor of π on the interaction part somehow – can’t tell where. So I hates it, but we’ll amend our result to:



This is the correct result, according to the literature I’ve seen. And as stated above, the time evolution of any initial state ψ0(x) would be given by,



**Time Development of ψ0(x) = δ(x-x0)**

The easiest two states to work out would be a position eigenstate, ψ0(x) = δ(x-x0). Then,



**Time Development of ψ0(x) = eikx**

And the next guy we’d like to try, which involves a bit more effort, is a momentum eigenstate ψ0(x) = eikx/√(2π),



Got a lot more integration to do. We’ll work this out term by term,



where we use ∫-∞∞exp[-(ax2 + bx + c)dx = √(π/a)∙exp(-b2/4a-c). And this continues to simplify to:



So,



This term is of course the evolution of a free particle. Next,



I guess I will try integration by parts. So I’ll need to have the anti-derivative of the emV0x´cos(kx´) term.



So filling this in,



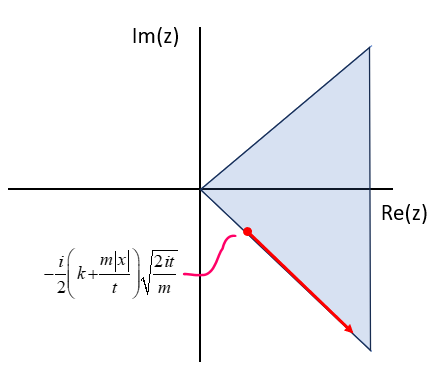
In preparation for these integrals, let’s look at:



Now let’s do an eighth-circle (so 45o) contour in the first quadrant of the complex plane. Since it encloses no singularities we can equate the integral along the real axis to the one along the eiπ/4 ray. And we have:



where the integral defining this erfc is along this contour (the triangle is the region within the RHS of the complex plane that the erfc converges),



So then,



and,



So, yuckily,



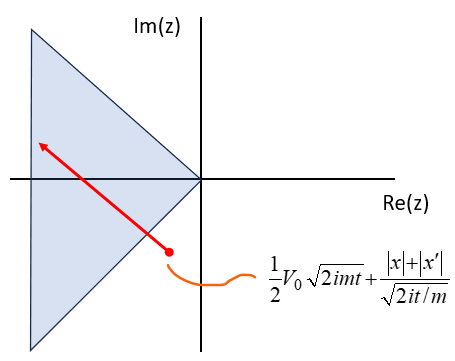
So,



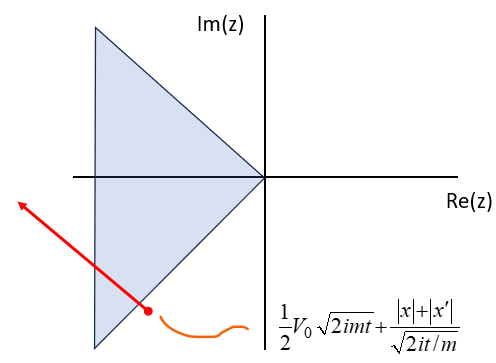
This is not very nice. Let’s consider long times, t. The erfc( )’s require some care. First up is this guy,



which we’ll recall was defined along this contour,



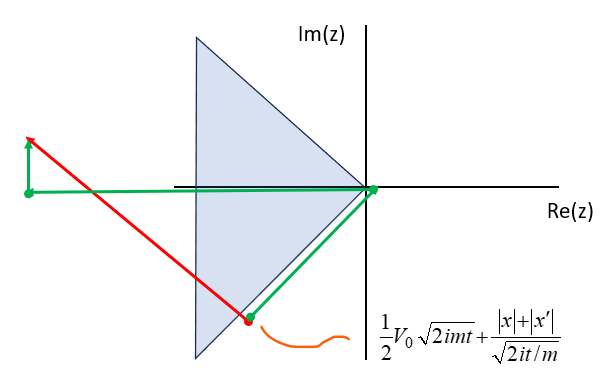
As we take t -> ∞, the contour shifts to this:



So we can say at least,



Not sure what this is in the limit. But we can work it out. We’ll take advantage of the fact that exp(-z2) has no poles in the complex plane, so integrals around a closed path give zero. So we can say the integral along the red contour is negative the integral along the green contour,



The integral along the green contour is, sans the vertical part which doesn’t matter because the integrand is negligible out there,



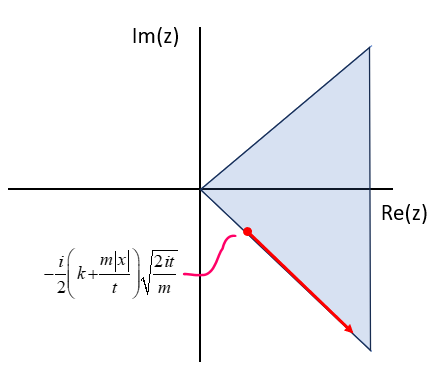
In large y limit, this goes to:



So I guess this means it is acceptable to approximate erfc(V0√(mit/2)) with the asymptotic approximation erfc(z) ~ exp(-z2)/z√π. Well. Now let’s look at the other one.



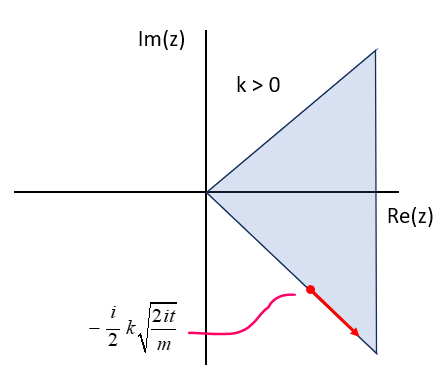
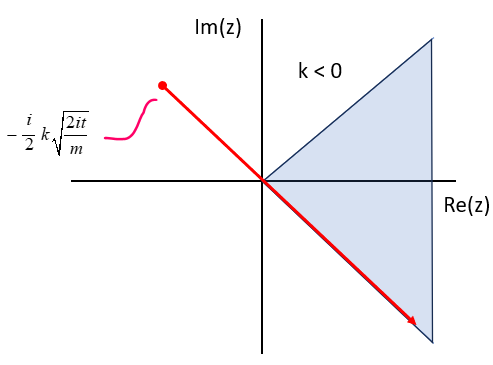
which was defined along the contour,



As t -> ∞, this will simplify to:



and the contour shifts to,

So as t -> ∞, we will get 0 for k > 0, and for k < 0,



Using general formula ∫-∞∞exp[-(ax2 + bx + c)dx = √(π/a)∙exp(-b2/4a-c). This is a formula worth remembering. So we have:



Now let’s fill these results into the boxed result, recalling our k is presumed positive (and so -k is negative),



So this is,



So,



And last we have:



We can break down the εb if we want,



So,



This is just the time-evolution of the bound eigenstate, with a prefactor accounting for the overlap of eikx with this state. So in the long time limit, we have:



So when our potential V(x) = V0δ(x) is turned on, our wave both reflects and transmits, and gets stuck, a little, in the bound state of the potential. We can interpret the first term as the incident wave. The second term contains the reflected and transmitted waves (x < 0, and x > 0 respectively – see scattering folder). The third term is the bound state illumination. The bound state term shouldn’t be here if V0 > 0, though I don’t know where in the prevoius 20 pages of analysis it would go away. We’ll recall that in the time-independent scattering case, we found that for a delta function potential barrier, the steady-state wavefunction was:



So one might think this should be the ratio of the ψ(2) and ψ(1) wavefunctions. But instead, it’s,



I’m guessing it’s off because not all of the wavefunction goes into the incident, reflected, transmitted waves; some of the wave is taken up by the bound state? Of course our prefactor is ‘bigger’ now, with the bound state. So does that make sense?